EE 2030 Linear Algebra Chi-chao Chao

◎ Vectors and Linear Combinations

$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ (column) vector

$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $\underline{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \Rightarrow \underline{v} + \underline{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix}$ vector addition

$2\underline{v} = \begin{bmatrix} 2v_1 \\ 2v_2 \end{bmatrix}$, $c\underline{v} = \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix}$ scalar multiplication

scalar

$c\underline{v} + d\underline{w} = \begin{bmatrix} cv_1 + dw_1 \\ cv_2 + dw_2 \end{bmatrix}$ linear combination of $\underline{v}$ and $\underline{w}$

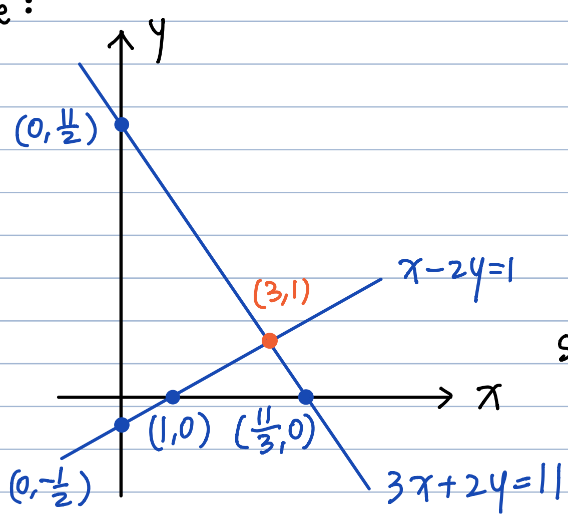
◎ Linear Equations

☆ Example

$$\begin{aligned} x - 2y &= 1 \\ 3x + 2y &= 11 \end{aligned} \qquad \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix} \qquad \boxed{A\underline{x} = \underline{b}}$$

matrix

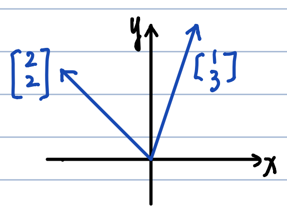
→ row picture :



solution : $x = 3$, $y = 1$

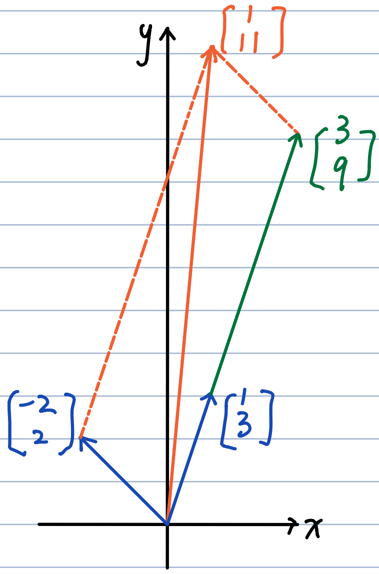
→ column picture :

$$x \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix} = \underline{b}$$



Find the linear combination of the vectors on the left side that equals the vector on the right.

$$3 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$



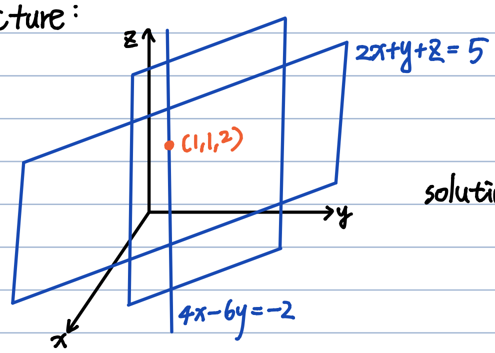
☆ Example

$$\begin{aligned} 2x+y+z &= 5 \\ 4x-6y &= -2 \\ -2x+7y+2z &= 9 \end{aligned}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

$$A\underline{x} = \underline{b}$$

→ row picture:

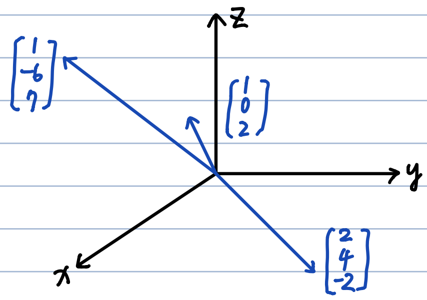


solution: $x=1$, $y=1$, $z=2$

→ column picture:

$$x\begin{bmatrix}2\\4\\-2\end{bmatrix} + y\begin{bmatrix}1\\-6\\7\end{bmatrix} + z\begin{bmatrix}1\\0\\2\end{bmatrix} = \begin{bmatrix}5\\-2\\9\end{bmatrix}$$

$$\begin{bmatrix}2\\4\\-2\end{bmatrix} + \begin{bmatrix}1\\-6\\7\end{bmatrix} + 2\begin{bmatrix}1\\0\\2\end{bmatrix} = \begin{bmatrix}5\\-2\\9\end{bmatrix}$$



☆ Example

$$x\begin{bmatrix}2\\4\\-2\end{bmatrix} + y\begin{bmatrix}1\\-6\\7\end{bmatrix} + z\begin{bmatrix}1\\0\\2\end{bmatrix} = \begin{bmatrix}3\\-2\\5\end{bmatrix}$$

$$\begin{bmatrix}2\\4\\-2\end{bmatrix} + \begin{bmatrix}1\\-6\\7\end{bmatrix} + 0\begin{bmatrix}1\\0\\2\end{bmatrix} = \begin{bmatrix}3\\-2\\5\end{bmatrix}$$

→ row picture: different three planes

→ column picture: the same three column vectors

☆ Question:

Can we solve $A\underline{x} = \underline{b}$ for every $\underline{b}$?

Do the linear combinations of the columns fill the 3-dimentional space?

→ Answer:

Yes for this A.

No when the three column vectors lie in the same place (singular case)

◎ Gaussian Elimination

*Carl Friedrich Gauss (1777-1855): German mathematician and physicist

$1+2+\cdots+100$ at age of 8.

☆ 例題

$\boxed{\text{*pivot}}$ n. 樞軸, 支點 / v. 在樞軸上轉動

pivot foot in basketball

pivot

$$\begin{aligned} 2x + 4y - 2z &= 2 \\ 4x + 9y - 3z &= 8 \\ -2x - 3y + 7z &= 10 \end{aligned}$$

×−2, ×1

$$\Rightarrow \begin{aligned} 2x + 4y - 2z &= 2 \\ 1y + z &= 4 \\ y + 5z &= 12 \end{aligned}$$

×−1

$$\Rightarrow \begin{aligned} 2x + 4y - 2z &= 2 \\ y + z &= 4 \\ 4z &= 8 \end{aligned}$$

pivots: 2, 1, 4

$z = 2$

$y + 2 = 4 \Rightarrow y = 2$

$2x + 8 - 4 = 2 \Rightarrow x = -1$

back substitution

$\Rightarrow x = -1,\ y = 2,\ z = 2$

$$\left[\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 5 & 12 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 4 & 8 \end{array}\right]$$

◎ In general, $\left[\begin{array}{c:c} A & \underline{b} \end{array}\right]$ $\quad A\underline{x} = \underline{b}$

$\Rightarrow \left[\begin{array}{c:c} U & \underline{c} \end{array}\right] \quad \Rightarrow U\underline{x} = \underline{c}$

U → upper-triangular matrix

◎ In order to solve the unknowns, pivots cannot be zero.

☆ When would the process break down?

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & \\ 2 & 2 & 5 & \\ 4 & 6 & 8 & \end{array}\right] \Rightarrow \left[\begin{array}{ccc:c} 1 & 1 & 1 & \\ 0 & 0 & 3 & \\ 0 & 2 & 4 & \end{array}\right] \Rightarrow \left[\begin{array}{ccc:c} 1 & 1 & 1 & \\ 0 & 2 & 4 & \\ 0 & 0 & 3 & \end{array}\right]$$ OK. (nonsingular case)

sometimes rows should be exchanged.

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & \\ 2 & 2 & 5 & \\ 4 & 4 & 8 & \end{array}\right] \Rightarrow \left[\begin{array}{ccc:c} 1 & 1 & 1 & \\ 0 & 0 & 3 & \\ 0 & 0 & 4 & \end{array}\right]$$ singular case

$$\begin{aligned} x + y + z &= c_1 \\ 3z &= c_2 \\ 4z &= c_3 \end{aligned}$$

These equations may be solvable or unsolvable.